Sovereign debt crises and cross-country assistance

Appendix

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September 2017

This appendix provides a more detailed solution of the model than provided in the main article, as well as an extension and additional discussion. Section 1 solves for the policy functions of countries \( r \) and \( s \). Section 2 provides more details on the proof of proposition 1. Section 3 extends the model by allowing domestic assistance. Section 4 offers a detailed discussion of the main results.

1 Optimal taxation and assistance

1.1 Optimal taxation and default choices for the risky country

The budget constraint of sovereign \( r \) is

\[
T^r + A = G^r + (1 - \theta) R^r b^r + \alpha^r \theta R^r b^r
\]  

(1)

Solving (1) for \( \theta \) and substituting the result into the resource constraint for consumers in \( r \), we get

\[
c^r = y^r - z(T^r) - T^r + (1 - \beta^r) b^r R^r - (1 + \kappa^r) (1 - \beta^r) \frac{G^r + b^r R^r - T^r - A}{1 - \alpha^r}
\]  

(2)

The government in \( r \) chooses \( T^r \) (and thus implicitly \( \theta \)) to maximize \( c^r \), subject to (1). From (2) we obtain the first-order condition for \( T^r \) when the marginal revenue is used to service debt:

\[
\frac{dc^r}{dT^r} = -z' (T^{r*}) - 1 + \frac{1 - \beta^r}{1 - \alpha^r} (1 + \kappa^r) = 0,
\]

or equivalently

\[
1 + z' (T^{r*}) = (1 - \beta^r) \frac{1 + \kappa^r}{1 - \alpha^r}
\]

(3)

We note that because \( z'' (\cdot) > 0 \), a necessary condition for any repayment is that \( \frac{dc^r}{dT^r}|_{T^r = G^r} > 0 \), or \((1 - \beta^r) (1 + \kappa^r) / (1 - \alpha^r) > 1 + z' (G^r)\). We assume this parameter restriction holds, otherwise \( r \) would always choose to default.
Equation (3) implicitly defines $r$’s “tax capacity” $T^r$: $r$ will never be willing (nor able to commit) to raise taxes beyond $T^r$. Country $r$’s optimal tax choice follows from combining (3) with the requirement that $\theta \in [0, 1]$ and the budget constraint. Note that absent assistance, $r$ will choose to repay fully if and only if

$$T^r \geq G^r + b^r R^r,$$

where the right hand side is expenditure when $\theta = 0$. Hence, there is an upper limit on $R^r_0$ at which $\theta = 0$ in the absence of bailout. For future reference we denote this interest rate by $\hat{R}$, where

$$\hat{R} \equiv \frac{T^r - G^r}{b^r}$$

On the other hand, in the absence of a bailout, $r$ will choose to default fully if and only if

$$T^r < G^r + \alpha^r b^r R^r,$$

where the right-hand side is government expenditure when $\theta = 1$.

It will be useful to express the tax schedule as a function of assistance $A$. Define

$$A \equiv G^r + \alpha^r b^r R^r - T^r,$$

$$\overline{A} \equiv G^r + b^r R^r - T^r,$$

Here $\overline{A}$ denotes the assistance needed for $r$ to repay fully when taxes are $T^r$. $A$ denotes the assistance needed for $r$ to cover its default costs and $G^r$ if default is full and taxes are set to $T^r$.

From (5), we see that if $A < A$ the bailout is so small that the tax capacity is insufficient to cover expenses even with a full default. In this case the government must raise $T^r > T^r$ to cover default costs. Conversely, if $A > \overline{A}$ we see from (4) that the bailout is so large that the risky government can repay fully with taxes below $T^r$. Finally, if $\overline{A} \leq A \leq \overline{A}$, the government in $r$ will set $T^r = T^r$ as implied by the first-order condition, and partially default. This can be summarized as follows:

$$T^r (A) = \begin{cases} 
G^r + \alpha^r b^r R^r - A & \text{if } A < A \\
T^r & \text{if } \overline{A} \leq A \leq \overline{A} \\
G^r + b^r R^r & \text{if } A > \overline{A}
\end{cases}$$

This shows that assistance fully crowds out taxes unless $\overline{A} \leq A \leq \overline{A}$, where taxes are $T^r$ independently of $A$. The three cases above are associated with the following default rates:

$$\theta (A) = \begin{cases} 
1 & \text{if } A < A \\
\frac{G^r + b^r R^r - T^r - A}{b^r R^r (1 - \alpha^r)} & \text{if } \overline{A} \leq A \leq \overline{A} \\
0 & \text{if } A > \overline{A}
\end{cases}$$
where $\theta$ in the intermediate case is calculated using $r$’s budget constraint. Notably, the effect of assistance on debt repayment may be calculated by differentiating (9) with respect to $A$:

$$
\theta' (A) = \begin{cases} 
-\frac{b_r R_r (1-\alpha_r)}{b_r R_r (1-\alpha_r)} & \text{if } A \leq A \leq \bar{A} \\
0 & \text{otherwise}
\end{cases}
$$

(10)

We see that a marginal increase in assistance either reduces the default incidence, or has no effect at all. The intuition is as follows. If $A < \underline{A}$ the bailout is so small that a marginal increase will not change $r$’s default decision: $r$ defaults fully anyway, and uses any assistance it receives to cut $T_r$. Conversely, if $A > \bar{A}$ the bailout is so large that a marginal change will again not change $r$’s default decision: $r$ already has sufficient funds to repay fully, and uses any additional assistance to cut $T_r$. Only if $A \leq A \leq \bar{A}$ will a marginal increase in $A$ influence the default decision. In this range, increasing the bailout does not affect the risky country’s tax choice which stays at $T_r = T_r^*$, but $r$ instead uses the additional funds to repay its creditors. Notably, because repayment prevents default costs $\alpha_r$, the amount repaid increases more than one-for-one with assistance.

1.2 Optimal taxation and assistance choices for the safe country

The safe country chooses $A$ (and thus $T^s$) to maximize $c^s$ subject to the government budget constraint. The marginal benefit of a tax-financed increase in $A$ is:

$$
\frac{dc^s}{dA} = -z'(T^s) - 1 - \beta^r R_r b'_r \theta' (A) - \kappa^s \beta^r b'_r R^s_r \theta' (A).
$$

(11)

From (10), $\theta' (A) = 0$ and hence $\frac{dc^s}{dA} < 0$ if $A < \underline{A}$ or $A > \bar{A}$. This implies that either $A = 0$ or $A \leq A \leq \bar{A}$. Consider setting $A \leq A \leq \bar{A}$. Substituting (10) into (11) yields

$$
\frac{dc^s}{dA} = \frac{\beta^r (1 + \kappa^s)}{(1 - \alpha^r)} - 1 - z'(T^s).
$$

(12)

From the properties of $z(\cdot)$ it follows that $\frac{dc^s}{dA} < 0$ for all $A$ if $\frac{\beta^r (1 + \kappa^s)}{(1 - \alpha^r)} < 1 + z'(G^s)$. Hence, we proceed under the assumption $\frac{\beta^r (1 + \kappa^s)}{(1 - \alpha^r)} > 1 + z'(G^s)$. The first-order condition for an internal assistance optimum is then

$$
1 + z'(T^{s*}) = \frac{\beta^r (1 + \kappa)}{(1 - \alpha^r)}.
$$

(13)

This condition implicitly defines $T^{s*}$, as a function of parameters. From $s$’s budget constraint, there will be a corresponding optimal assistance level $A^*$:

$$
1 + z'(A^* + G^s) = \frac{\beta^r (1 + \kappa)}{(1 - \alpha^r)}
$$
In summary, we have shown that the optimal assistance policy is characterized by

\[ A = \begin{cases} \overline{A} & \text{if } A^* > \overline{A} \\ A^* & \text{if } A \leq A^* \leq \overline{A} \\ 0 & \text{if } A^* < A \end{cases} , \] (14)

where \( \overline{A} \) and \( \overline{A} \) are given in equations (6) and (7), respectively. The first part of this schedule states that assistance will never exceed \( \overline{A} \), as any assistance above this level will not find its way back to the safe country’s residents. The next part describes an internal optimum, where the marginal deadweight costs of taxes equal the marginal gains of preventing default as expressed in (13). This occurs when the recipient country uses assistance to repay debt. The final part shows that assistance below a certain threshold is not worth providing, as debtors then will not repay anything anyhow.

1.3 Resources for repayment

The risky country defaults whenever

\[ T^r + A < G^r + b^r R^r \]

As before, there are three scenarios to consider under default. If \( A^* < \overline{A} \), implying \( T^{r*} + A^* < G^r + \alpha^r b^r R^r \), the safe country sets \( A = 0 \). There is a full default, \( \theta = 1 \), and \( r \) raises taxes to cover default costs only. Second, if \( A^* > \overline{A} \), implying \( T^{r*} + A^* > G^r + b^r R^r \), the \( A^* \) is so large relative to the amount outstanding, that \( r \) has more funds than needed to repay fully. Realizing this, \( s \) will not offer \( A^* \) but limits the bailout to \( A = \overline{A} = G^r + b^r R^r + \alpha^r T^{r*} \), so that \( \theta = 0 \) if \( T^r = T^{r*} \). In response, \( r \) sets taxes to \( T^{r*} \). The overall funds available to \( r \) equal the amount needed for expenditure and debt repayment. Third, if \( \overline{A} \leq A^* \leq \overline{A} \), implying \( G^r + \alpha^r b^r R^r \leq T^{r*} + A^* \leq G^r + b^r R^r \), the safe country sets \( A = A^* \) while \( r \) sets \( T^r = T^{r*} \).

To summarize, the funds spent by \( r \) are given by

\[ T^r + A = \begin{cases} G^r + \alpha^r b^r R^r & \text{if } T^{r*} + A^* < G^r + \alpha^r b^r R^r \\ G^r + b^r R^r & \text{if } T^{r*} + A^* > G^r + b^r R^r \\ T^{r*} + A^* & \text{otherwise} \end{cases} \] (15)

We see that the repayment funds depend on the interest rate \( R^r_b \). Let us define

\[ R^r = \frac{T^{r*} - G^r + A^*}{b^r} \] (16)

\[ \overline{R}^r = \frac{T^{r*} - G^r + A^*}{\alpha^r b^r} \] (17)
The total funds spent by country $r$ can therefore be expressed as

\[ T^r + A = \begin{cases} 
  G^r + b^r R^r & \text{if } R^r < \overline{R}^r \\
  G^r + \alpha^r b^r R^r & \text{if } R^r > \overline{R}^r \\
  T^{r*} + A^* & \text{otherwise}
\end{cases} \]  

(18)

If $R^r < \overline{R}^r$, country $r$ repays in full, and assistance and tax revenues are used to finance public goods and servicing outstanding debt. Conversely, if $R^r > \overline{R}^r$, country $r$ does not repay anything, and hence incurs default costs $\alpha^r$ for each unit of outstanding debt. Available resources are then spent on public goods and default costs, but not repayment of debt. Finally, in the intermediate region there is partial default ($\theta > 0$), while total tax revenues and assistance are at capacity. Since neither country is willing to increase its expenditure voluntarily, repayment does not increase with $R^r$. Instead, higher costs of debt service just increase the default rate.

1.4 Equilibrium pricing of debt

Arbitrage requires

\[ R = R^r [1 - E(\theta)] \]  

(19)

Rearranging equation (19) and imposing perfect foresight, we get $\theta R^r = R^r - R$, which can be substituted into $r$’s government budget constraint to obtain

\[ T^r + A = G^r + (1 - \theta^r) b^r R + \theta^r b^r R^r \]  

(20)

This is the 2-country equivalent to what Calvo (1988) calls the “consistency condition”. It gives levels of $T^r + A$ that are consistent with the no-arbitrage condition for $R^r$ determined in period 0.

2 Proof of Proposition 1

Proposition 1 Existence of equilibria:

1. If $T^{r*} + A^* < G^r + R b^r$, no equilibrium exists.

2. If $T^{r*} + A^* = G^r + R b^r$, a unique equilibrium exists, with $R^r = R$ and $\theta = 0$.

3. If $T^{r*} + A^* > G^r + R b^r$, two equilibria exist: one with $R^r = R$ and $\theta = 0$, and one with $R^r = R^{r*} = \frac{T^{r*} + A^* - G^r - (1 - \alpha^r) R b^r}{\alpha^r b^r}$ and $\theta = \frac{G^r + b^r R^{r*} - T^{r*} - A^*}{b^r R^{r*}(1 - \alpha^r)}$.

Proof Part 1: (1) implies $T^r + A = G^r + [(1 - \theta) R^r + \alpha^r \theta R^r] b^r$, while (19) implies $R < (1 - \theta) R^r + \alpha^r \theta R^r$. Hence, if $T^{r*} + A^* < G^r + R b^r$, there exists no interest rate compatible with both (1) and (19). This proves part 1.
Part 2: Assume $T^r + A = T^{r*} + A^*$. Then both (1) and (19) hold if and only if $R^r = R$ and $\theta = 0$. (15) then validates $T^r + A = T^{r*} + A^*$. Hence, an equilibrium with $\{R^r = R, \theta = 0\}$ exists. (1), (19) and (15) cannot simultaneously hold for any other $R^r$.

Part 3: Assume $T^r + A = T^{r*} + A^*$. Then both (1) and (19) hold if and only if $R^r = R^{r*}$ and $\theta = \theta^*$. $R^r = R^{r*}$ implies $G^r + \alpha^r R^r b^r = T^{r*} + A^* - (1 - \alpha^r) Rb^r$, and hence $G^r + \alpha^r R^r b^r \leq T^{r*} + A^*$. $R^r = R^{r*}$ also implies $G^r + R^r b^r = G^r + \frac{T^{r*} + A^* - (1 - \alpha^r) Rb^r}{\alpha^r} = T^{r*} + A^* + \frac{1 - \alpha^r}{\alpha^r} (T^{r*} + A^* - G^r - Rb^r)$. From $T^{r*} + A^* > G^r + Rb^r$, it then follows that $G^r + R^r b^r > T^{r*} + A^*$. Hence, $G^r + \alpha^r R^r b^r < T^{r*} + A^* < G^r + R^r b^r$, and (15) validates $T^r + A = T^{r*} + A^*$. Thus, an equilibrium with $\{R^r = R^{r*}, \theta = \theta^*\}$ exists.

If $R^r = R$, $T^{r*} + A^* > G^r + Rb^r$ and (15) imply $T^r + A = G^r + Rb^r$. (1) and (19) then hold with $\theta = 0$. Hence, an equilibrium with $\{R^r = R, \theta = 0\}$ exists. (1), (19) and (15) cannot simultaneously hold for any other $R^r$.

3 Cross-country versus domestic assistance

One might question why a non-defaulting country would choose to assist rather than to just save its own creditors. Extending our framework to allow for this possibility is relatively straightforward, by allowing the safe country to issue domestic assistance $a^s$ to its own residents only, in order to compensate them from losses on $r$’s default. The government budget constraint in country $s$ becomes:

$$T^s - G^s = A + a^s$$

and the resource constraint on consumption becomes:

$$c^s = y^s + a^s - z(T^s) - T^s + \beta^r (1 - \theta) R^r b^r - \kappa [\beta^r \theta b^r R^r - a^s]^+. $$

Here the spillover-costs from $r$’s default depend on losses suffered by domestic households, net of transfers $a^s$, and $[\beta^r \theta b^r R^r - a^s]^+ \equiv \min \{0, \beta^r \theta b^r R^r - a^s\}$. The following proposition summarizes how $s$ might still find it optimal to provide cross-country assistance $A$ as studied above:

**Proposition 2** Cross-country assistance versus domestic bailouts:

1. If $\beta^r + \alpha^r > 1$ and $\hat{R} \leq R^r \leq \overline{R}$, then $s$ assists $r$ rather than its own residents.

2. Otherwise, $s$ does not assist $r$, but assists its own residents instead.

**Proof** The government of the safe country maximizes $c^s$ subject to its budget constraint. The marginal effect of raising taxes to finance domestic transfers is

$$\frac{dc^s}{da^s} = -z'(T^s) + \kappa$$
The safe country will prefer to assist the risky country rather than to bail out its own citizens directly, if and only if \( \frac{d\alpha_s}{dA} - \frac{dc_s}{d\alpha_s} > 0 \). From (12), this can be expressed as

\[
\frac{dc_s}{dA} - \frac{dc_s}{d\alpha_s} = -\theta'(A) \beta r^r b^r T_r r^r - (1 + \kappa) - 1 - \kappa,
\]

Recall from (10) that \( \theta'(A) = -\frac{1}{b^r R_r (1 - \alpha^r)} \) if the interest rate is in the region \( \hat{R} \leq R^r \leq \bar{R}^r \), while \( \theta'(A) = 0 \) otherwise. It follows that

\[
\frac{\partial c_s}{\partial A} - \frac{\partial c_s}{\partial \alpha_s} = \begin{cases} 
\beta r^r \frac{1 + \kappa}{1 - \alpha^r} - (1 + \kappa) & \text{if } \hat{R} \leq R^r \leq \bar{R}^r \\
0 & \text{otherwise}
\end{cases}
\]

Hence, \( \frac{\partial c_s}{\partial A} - \frac{\partial c_s}{\partial \alpha_s} > 0 \) if and only if \( \beta r^r + \alpha^r > 1 \).■

Intuitively, if assistance reduces \( \theta \), it prevents default costs (\( \alpha^r \)) from materializing. This frees up resources. Hence, assistance stimulates repayment more than one-for-one through a “default cost multiplier”. Moreover, country \( s \)'s valuation of \( r \)'s repayments increases with its residents' claims, \( \beta^r \). The condition \( \hat{R} \leq R^r \leq \bar{R}^r \) ensures that the receiving country does not use all assistance to cut taxes. Hence, when \( \beta^r + \alpha^r > 1 \), all the results from our main analysis go through.

4 Additional discussion

Figure 1 illustrates the possible equilibria. Total funding for \( r \), \( T^r + A \), is a function of \( R^r \). The points \((R^r, T^{rs} + A^s)\) and \((\bar{R}^r, T^{rs} + A^s)\) mark the kinks in the \( T^r + A \) schedule. The market consistency condition (20) is drawn for 2 alternative scenarios satisfying the criteria in part 3 of the proposition. This curve is increasing in \( R^r \), with slope \( \alpha^r b^r \) implying that it runs parallel to the upward segment of the funding curve. Because repayment is non-negative, the consistency condition is only relevant when \( R \leq R^r \). Possible equilibria are intersections of \( T^r + A \) with the consistency condition. In Figure 1, this occurs first at \( E^0 \) where all debt is repaid fully and investors therefore are happy to hold debt at the same interest rate as that paid on the safe alternative, \( R \). The second possible equilibrium occurs at \( E^1 \), where \( r \) defaults partly (\( 0 < \theta < 1 \)) and the market interest rate on \( r \)'s debt is greater than \( R \) as investors require compensation for the haircut. Hence, assistance by \( s \) does not prevent the possibility of a self-fulfilling default. Compared to the partial-default equilibrium without assistance, which would occur at the intersection of the solid curve and the consistency condition (\( E^{1a} \) in the figure), we see that all assistance achieves in this situation is to raise the equilibrium interest rate that is consistent with default. Furthermore, since the interest rate is higher, condition (20) implies that the default rate in this equilibrium necessarily also is higher than without assistance.
Figure 1: Equilibrium

Note: Taxes in the risky country $T^r$ (solid line) and assistance $A$ (dashed line) as functions of the interest rate $R^r$. The dotted line gives the interest rate required in the market as function of total repayment. Possible equilibria in the different scenarios explained in the main text are indicated by $(E^0, E^{1a}, E^1)$ for a country with relatively low debt, and $(P^0, P^1)$ for a country with relatively high debt.

With different initial conditions in terms of $G^r$ or $b^r$, the impact of assistance might change. For instance, if $r$’s initial debt is sufficiently high, $r$ alone will not be able to service its debt even at the lowest possible interest rate $R$. Here assistance may play a role, by preventing what would otherwise have been a certain default. Such a scenario is illustrated by the consistency condition running through points $P^0$ and $P^1$ in the figure. Absent assistance, there exists no equilibrium interest rate such that $r$ is able to repay investors. Assistance, however, facilitates equilibria with repayment, since investors realize that even though $r$ can never repay its debt alone, there are interest rates such that $s$ will find it in its own best interest to provide sufficient assistance for debt to be repaid. Therefore, investors are willing to hold this debt. Again, there are two possible equilibria, $P^0$ with full repayment ($\theta = 0$) and $P^1$ with partial repayment ($0 < \theta < 1$),